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Perturbation to Mei Symmetry and Generalized Mei Adiabatic Invariants for Birkhoffian Systems

Ming-Jiang Zhang · Jian-Hui Fang · Kai Lu

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Abstract This paper investigates the perturbation to Mei symmetry for Birkhoffian systems. The criterion equation of the perturbation to Mei symmetry is established. The condition for existence of generalized Mei adiabatic invariant induced directly from the perturbation to Mei symmetry is obtained, and its form is presented. Finally, an example is discussed to further illustrate the application of the results.

Keywords Analytical mechanics · Birkhoffian system · Mei symmetry · Perturbation · Adiabatic invariant

1 Introduction

Symmetry is an important and universal physical nature. Even tiny changes in symmetries, so-called perturbation to symmetries, are of great importance for physical systems [1–3]. Pioneered in this area, Burgers proposed adiabatic invariants for a special kind of Hamiltonian systems [4]. The adiabatic invariants play an important role in the research on the quasi-integrability of mechanical systems. At present, more and more attention has been paid to this research field, and many important results have been obtained [5–10]. But most of these researches have focused on perturbation to Noether symmetry or Lie symmetry, and the adiabatic invariants obtained by these researches belong to the types of Noether [5–7] and Hojman [8–10]. In the beginning of this century, Mei presented a new theory of symmetries, form invariance [11, 12], which enriches the symmetry theories of mechanical systems.

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M.-J. Zhang (✉) · J.-H. Fang · K. Lu
College of Physics Science and Technology, China University of Petroleum, Dongying 257061, People's Republic of China
e-mail: zhangmj517@163.com

J.-H. Fang
e-mail: fangjh@upc.edu.cn

Mei's form invariance is also called Mei symmetry [13, 14], which means that the dynamical functions in equations of motion still satisfy the equations' primary form after some infinitesimal transformations. Recently, we have studied the perturbation to Mei symmetry for mechanical systems in phase space, and obtained Mei adiabatic invariants of the systems [15]. Fang et al. discussed a new type of exact invariants or generalized Mei exact invariants induced directly from Mei symmetry of Lagrange systems and Hamiltonian systems respectively [16, 17]. And [18] studied a generalized Mei exact invariant induced directly from Mei symmetry of Birkhoffian system. However, new types of adiabatic invariants or generalized Mei adiabatic invariants have rarely been studied so far.

The Birkhoffian dynamics is more general than the Hamiltonian mechanics and it is extensively applied to the fields of modern physics. In 1927, G.D. Birkhoff made primary studies on Birkhoffian dynamics [19]. Then the researches on Birkhoffian dynamics become more and more active [20–23]. The research on perturbation to symmetries and adiabatic invariants for Birkhoffian systems has great important significance. Since 2000, some works have been done to discuss the perturbation to symmetries and adiabatic invariants for Birkhoffian systems [24–26]. We have studied the perturbation to Lie-Mei symmetry for Birkhoffian systems, and obtained Mei adiabatic invariants of the systems [27].

In this paper, we further study the perturbation to Mei symmetry for Birkhoffian systems, and a new type of adiabatic invariant induced directly from the perturbation to Mei symmetry is obtained. This new type of adiabatic invariant can be called generalized Mei adiabatic invariant.

2 Birkhoffian Equations

The Birkhoffian equations can be written as

$$\Omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial t} = 0, \quad (\mu, \nu = 1, \ldots, 2n), \tag{1}$$

where $B = B(t, \boldsymbol{a})$ is called a Birkhoffian, $R_{\mu} = R_{\mu}(t, \boldsymbol{a})$ are called Birkhoffian functions, a^{μ} are variables, and

$$\Omega_{\mu\nu} = \frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}. \tag{2}$$

The system described by (1) is called a Birkhoffian system. Suppose the system is non-singular, that is

$$\det(\Omega_{\mu\nu}) \neq 0, \tag{3}$$

then (1) can be expressed as

$$\dot{a}^{\mu} = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right), \tag{4}$$

where

$$\Omega^{\mu\nu} = \left[\left(\frac{\partial R_{\beta}}{\partial a^{\alpha}} - \frac{\partial R_{\alpha}}{\partial a^{\beta}}\right)^{-1}\right]^{\mu\nu}. \tag{5}$$

Expanding (4), we obtain

$$\dot{a}^{\mu} = h_{\mu}(t, \boldsymbol{a}). \tag{6}$$

3 Mei Symmetry and Generalized Mei Exact Invariants

Introduce the infinitesimal transformations with respect to time t and variables a^μ

$$t^* = t + \varepsilon\tau^0(t, \boldsymbol{a}), \quad a^{\mu *}(t^*) = a^\mu(t) + \varepsilon\xi_\mu^0(t, \boldsymbol{a}), \tag{7}$$

where ε is an infinitesimal parameter, while τ^0 and ξ_μ^0 are infinitesimal generators. Then the Birkhoffian functions $R_\mu = R_\mu(t, \boldsymbol{a})$ become $R_\mu^* = R_\mu(t^*, \boldsymbol{a}^*)$, and the Birkhoffian $B = B(t, \boldsymbol{a})$ becomes $B^* = B(t^*, \boldsymbol{a}^*)$ under the infinitesimal transformations (7). Expanding R_μ^* and B^*, we have

$$\begin{aligned} R_\mu^* &= R_\mu(t^*, \boldsymbol{a}^*) = R_\mu(t, \boldsymbol{a}) + \varepsilon X_0^0(R_\mu) + O(\varepsilon^2), \\ B^* &= B(t^*, \boldsymbol{a}^*) = B(t, \boldsymbol{a}) + \varepsilon X_0^0(B) + O(\varepsilon^2), \end{aligned} \tag{8}$$

where

$$X_0^{(0)} = \tau^0 \frac{\partial}{\partial t} + \xi_\mu^0 \frac{\partial}{\partial a^\mu}. \tag{9}$$

Definition 1 After the Birkhoffian functions $R_\mu = R_\mu(t, \boldsymbol{a})$ and the Birkhoffian $B = B(t, \boldsymbol{a})$ being replaced by the transformed Birkhoffian functions $R_\mu^* = R_\mu(t^*, \boldsymbol{a}^*)$ and the transformed Birkhoffian $B^* = B(t^*, \boldsymbol{a}^*)$ respectively, if the form of equation (1) is invariant, that is

$$\left(\frac{\partial R_\nu^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial t} = 0, \quad (\mu, \nu = 1, \ldots, 2n), \tag{10}$$

the invariance is called Mei symmetry of the Birkhoffian system.

Substituting (6) and (8) into (10), ignoring ε^2 and the higher order infinitesimal terms, and considering (1), we obtain the following criterion of Mei symmetry for the Birkhoffian system.

Criterion 1 For the Birkhoffian system (1), if the infinitesimal generators τ^0 and ξ_μ^0 satisfy

$$\left[\frac{\partial X_0^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_0^{(0)}(R_\mu)}{\partial a^\nu}\right] h_\nu - \frac{\partial X_0^{(0)}(B)}{\partial a^\mu} - \frac{\partial X_0^{(0)}(R_\mu)}{\partial t} = 0, \tag{11}$$

the corresponding invariance is Mei symmetry of the system.

Equation (11) is called criterion equation of Mei symmetry for the Birkhoffian system. Under certain condition, the Mei symmetry will lead to a generalized Mei exact invariant. The condition of existence and the form of the generalized Mei exact invariant are given in the following proposition.

Proposition 1 *For the Birkhoffian system* (1), *if the infinitesimal generators* τ^0, ξ_μ^0 *of Mei symmetry and the gauge function* $G_M^0 = G_M^0(t, \mathbf{a})$ *satisfy the following structure equation*

$$\begin{aligned} &\left[f \frac{\partial X_0^{(0)}(R_\mu)}{\partial t} + \xi_\nu^0 \frac{\partial X_0^{(0)}(R_\mu)}{\partial a^\nu}\right] h_\mu - \left[f \frac{\partial X_0^{(0)}(B)}{\partial t} + \xi_\nu^0 \frac{\partial X_0^{(0)}(B)}{\partial a^\nu}\right] \\ &\quad - X_0^{(0)}(B) \frac{\bar{\mathrm{d}} f}{\mathrm{d}t} + X_0^{(0)}(R_\mu) \frac{\bar{\mathrm{d}}\xi_\mu^0}{\mathrm{d}t} + \frac{\bar{\mathrm{d}} G_M^0}{\mathrm{d}t} = 0, \end{aligned} \tag{12}$$

the Mei symmetry of the system can lead to a generalized Mei exact invariant [18]

$$I_{M0} = X_0^{(0)}(R_\mu)\xi_\mu^0 - X_0^{(0)}(B) f + G_M^0 = \text{const}, \tag{13}$$

where $f = f(t, \mathbf{a})$ *is an arbitrary function to make the gauge function exist*, *which can be called coordination function*. *And in* (12),

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + h_\mu \frac{\partial}{\partial a^\mu}. \tag{14}$$

If we let $f = \tau^0$, (12) *and* (13) *respectively become the structure equation and the Mei exact invariant in* [23].

4 Perturbation to Mei Symmetry and Generalized Mei Adiabatic Invariants

Small forces acting on the Birkhoffian system may result in a small change in its symmetries, defined as symmetrical perturbation. The exact invariant associated with the symmetries, under a corresponding change, is an adiabatic invariant. In analytical mechanics, we study perturbation to symmetries and adiabatic invariants of mechanical systems based on the concept of higher order adiabatic invariant. Now, we present the concept of higher order adiabatic invariant of the Birkhoffian system.

Definition 2 If $I_z(t, \boldsymbol{a}, \varepsilon)$ is a physical quantity including infinitesimal parameter ε in which the highest power is z in a Birkhoffian system, and its derivative with respect to time t is directly proportional to ε^{z+1}, I_z is called a zth order adiabatic invariant of the Birkhoffian system.

Suppose the Birkhoffian system (1) is perturbed by small quantities $\varepsilon W_\mu = \varepsilon W_\mu(t, \boldsymbol{a})$, then the equations of motion for the system become

$$\Omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = \varepsilon W_\mu. \tag{15}$$

From (1) and (6), (15) can be also written in the form

$$\dot{a}^\mu = h_\mu + \varepsilon \Omega^{\mu\nu} W_\nu. \tag{16}$$

Due to the action of εW_μ, the original symmetries and invariants of the system may vary. Assume that the variation is a small perturbation based on the symmetrical transformations of the system without perturbation, $\tau(t, \boldsymbol{a})$ and $\xi_\mu(t, \boldsymbol{a})$ express the generators of infinitesimal transformations after being perturbed, then

$$\tau = \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots, \qquad \xi_\mu = \xi_\mu^0 + \varepsilon\xi_\mu^1 + \varepsilon^2\xi_\mu^2 + \cdots. \tag{17}$$

And the infinitesimal transformations become

$$t^* = t + \varepsilon\tau(t, \boldsymbol{a}), \quad a^{\mu*}(t^*) = a^\mu(t) + \varepsilon\xi_\mu(t, \boldsymbol{a}). \tag{18}$$

According to the Mei symmetry theory, the criterion equation of the Mei symmetry for the system perturbed by small quantities εW_μ is

$$\left[\frac{\partial X^{(0)}(R_\nu)}{\partial a^\mu}-\frac{\partial X^{(0)}(R_\mu)}{\partial a^\nu}\right](h_\nu+\varepsilon\Omega^{\nu\mu}W_\mu)-\frac{\partial X^{(0)}(B)}{\partial a^\mu}-\frac{\partial X^{(0)}(R_\mu)}{\partial t}-\varepsilon X^{(0)}(W_\mu)=0, \tag{19}$$

where

$$X^{(0)}=\tau\frac{\partial}{\partial t}+\xi_\mu\frac{\partial}{\partial a^\mu}. \tag{20}$$

Substituting (17) into (20), we obtain

$$X^{(0)}=\varepsilon^m X_m^{(0)},\quad (m=0,1,\dots,z), \tag{21}$$

where

$$X_m^{(0)}=\tau^m\frac{\partial}{\partial t}+\xi_\mu^m\frac{\partial}{\partial a^\mu}. \tag{22}$$

Substituting (17) into (19), noticing (20)–(22), and making the coefficients of ε^m equal, we obtain

$$\left[\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu}-\frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\right]h_\nu+\left[\frac{\partial X_{m-1}^{(0)}(R_\nu)}{\partial a^\mu}-\frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\right]\Omega^{\nu\mu}W_\mu$$
$$-\frac{\partial X_m^{(0)}(B)}{\partial a^\mu}-\frac{\partial X_m^{(0)}(R_\mu)}{\partial t}-X_{m-1}^{(0)}(W_\mu)=0. \tag{23}$$

Then we can obtain the following criterion.

Criterion 2 For the Birkhoffian system (1) perturbed by small quantities εW_μ, if the infinitesimal generators $\tau^m(t,\mathbf{a})$ and $\xi_\mu^m(t,\mathbf{a})$ satisfy (23), the corresponding variation of the Mei symmetry is perturbation to Mei symmetry.

Equation (23) can be called criterion equation of perturbation to Mei symmetry for the Birkhoffian system. Meanwhile, after the system being perturbed, the primary structure equation varies. Contrasting with (12), we construct the following structure equation

$$\left[f\frac{\partial X^{(0)}(R_\mu)}{\partial t}+\xi_\nu^0\frac{\partial X^{(0)}(R_\mu)}{\partial a^\nu}\right](h_\mu+\varepsilon\Omega^{\mu\nu}W_\nu)$$
$$-\left[f\frac{\partial X^{(0)}(B)}{\partial t}+\xi_\nu^0\frac{\partial X^{(0)}(B)}{\partial a^\nu}\right]-X^{(0)}(B)\frac{\tilde{d}f}{dt}$$
$$+X^{(0)}(R_\mu)\frac{\tilde{d}\xi_\mu^0}{dt}-\varepsilon X^{(0)}(W_\mu)[\xi_\mu^0-(h_\mu+\varepsilon\Omega^{\mu\nu}W_\nu)f]+\frac{\tilde{d}G_M}{dt}=0, \tag{24}$$

where

$$\frac{\tilde{d}}{dt}=\frac{\partial}{\partial t}+(h_\mu+\varepsilon\Omega^{\mu\nu}W_\nu)\frac{\partial}{\partial a^\mu}, \tag{25}$$

and $G_M=G_M(t,\boldsymbol{a})$ is a gauge function. After being perturbed, the gauge function comes into

$$G_M=G_M^0+\varepsilon G_M^1+\varepsilon^2G_M^2+\cdots. \tag{26}$$

In order to use (25) expediently, we write it down simply as

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t} + \varepsilon \Omega^{\mu\nu} W_\nu \frac{\partial}{\partial a^\mu}. \tag{27}$$

Here $\bar{\mathrm{d}}/\mathrm{d}t$ has the same form as that in (13).

Substituting (17) and (27) into (24), noticing (20)–(22), and making the coefficients of ε^m equal, we obtain

$$\begin{aligned}
&\left[f \frac{\partial X_m^{(0)}(R_\mu)}{\partial t} + \xi_\nu^0 \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\right] h_\mu + \left[f \frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial t} + \xi_\nu^0 \frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\right] \Omega^{\mu\nu} W_\nu \\
&\quad - \left[f \frac{\partial X_m^{(0)}(B)}{\partial t} + \xi_\nu^0 \frac{\partial X_m^{(0)}(B)}{\partial a^\nu}\right] - X_m^{(0)}(B) \frac{\bar{\mathrm{d}} f}{\mathrm{d}t} - X_{m-1}^{(0)}(B) \Omega^{\alpha\beta} W_\beta \frac{\partial f}{\partial a^\alpha} + X_m^{(0)}(R_\mu) \frac{\bar{\mathrm{d}} \xi_\mu^0}{\mathrm{d}t} \\
&\quad + X_{m-1}^{(0)}(R_\mu) \Omega^{\alpha\beta} W_\beta \frac{\partial \xi_\mu^0}{\partial a^\alpha} - X_{m-1}^{(0)}(W_\mu)(\xi_\mu^0 - h_\mu f) + X_{m-2}^{(0)}(W_\mu) \Omega^{\mu\nu} W_\nu f \\
&\quad + \frac{\bar{\mathrm{d}} G_M^m}{\mathrm{d}t} + \Omega^{\alpha\beta} W_\beta \frac{\partial G_M^{m-1}}{\partial a^\alpha} = 0.
\end{aligned} \tag{28}$$

In (23) and (28) the terms with the minimum order in τ, ξ_μ, and G_M are τ^0, ξ_μ^0 and G_M^0 respectively. When $m = 0$, we note that $\tau^{m-2} = \xi_\mu^{m-2} = \tau^{m-1} = \xi_\mu^{m-1} = G_M^{m-1} = 0$ holds, then (23) and (28) will turn into (11) and (12) respectively. Also, when $m = 1$, we note that $\tau^{m-2} = \xi_\mu^{m-2} = 0$ holds.

Proposition 2 *For the Birkhoffian system* (1) *perturbed by small quantities* εW_μ, *if the infinitesimal generators* $\tau^m(t, \mathbf{a})$, $\xi_\mu^m(t, \mathbf{a})$ *and the gauge function* $G_M^m = G_M^m(t, \mathbf{a})$ *satisfy* (23) *and the structure equation* (28), *the perturbation to Mei symmetry of the system will lead to a zth order adiabatic invariant*

$$I_{Mz} = \varepsilon^m [X_m^{(0)}(R_\mu) \xi_\mu^0 - X_m^{(0)}(B) f + G_M^m], \quad (m = 0, 1, \ldots, z). \tag{29}$$

Proof Taking the derivative of I_{Mz} with respect to t, we have

$$\begin{aligned}
\frac{\tilde{\mathrm{d}} I_{Mz}}{\mathrm{d}t} &= \varepsilon^m \left\{ X_m^{(0)}(R_\mu) \frac{\tilde{\mathrm{d}} \xi_\mu^0}{\mathrm{d}t} + \left[\frac{\partial X_m^{(0)}(R_\mu)}{\partial t} + \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}(h_\nu + \varepsilon \Omega^{\nu\mu} W_\mu)\right] \xi_\mu^0 - X_m^{(0)}(B) \frac{\tilde{\mathrm{d}} f}{\mathrm{d}t} \right. \\
&\quad \left. - \left[\frac{\partial X_m^{(0)}(B)}{\partial t} + \frac{\partial X_m^{(0)}(B)}{\partial a^\nu}(h_\nu + \varepsilon \Omega^{\nu\mu} W_\mu)\right] f + \frac{\tilde{\mathrm{d}} G_M^m}{\mathrm{d}t} \right\} \\
&= \varepsilon^m \left\{ X_m^{(0)}(R_\mu) \frac{\tilde{\mathrm{d}} \xi_\mu^0}{\mathrm{d}t} - X_m^{(0)}(B) \frac{\tilde{\mathrm{d}} f}{\mathrm{d}t} + \frac{\partial X_m^{(0)}(R_\mu)}{\partial t}(h_\mu + \varepsilon \Omega^{\mu\nu} W_\nu) f - \frac{\partial X_m^{(0)}(B)}{\partial t} f \right. \\
&\quad + \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}(h_\mu + \varepsilon \Omega^{\mu\nu} W_\nu) \xi_\nu^0 - \frac{\partial X_m^{(0)}(B)}{\partial a^\nu} \xi_\nu^0 \\
&\quad - \left[\frac{\partial X_m^{(0)}(R_\mu)}{\partial t} + \frac{\partial X_m^{(0)}(B)}{\partial a^\mu}\right](h_\mu + \varepsilon \Omega^{\mu\nu} W_\nu) f \\
&\quad - \left[\left(\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\right)(h_\nu + \varepsilon \Omega^{\nu\mu} W_\mu) \right.
\end{aligned} \tag{30}$$

$$- \frac{\partial X_m^{(0)}(B)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial t}\Big]\xi_\mu^0 + \frac{\tilde{\mathrm{d}}G_M^m}{\mathrm{d}t}\Big\}$$

$$= \varepsilon^m \Big\{ X_m^{(0)}(R_\mu)\frac{\bar{\mathrm{d}}\xi_\mu^0}{\mathrm{d}t} + X_m^{(0)}(R_\mu)\varepsilon\Omega^{\alpha\beta}W_\beta \frac{\partial \xi_\mu^0}{\partial a^\alpha} - X_m^{(0)}(B)\frac{\bar{\mathrm{d}}f}{\mathrm{d}t} - X_m^{(0)}(B)\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial f}{\partial a^\alpha}$$

$$+ \frac{\partial X_m^{(0)}(R_\mu)}{\partial t}(h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu)f - \frac{\partial X_m^{(0)}(B)}{\partial t} f + \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}(h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu)\xi_\nu^0$$

$$- \frac{\partial X_m^{(0)}(B)}{\partial a^\nu}\xi_\nu^0 - \Big[\frac{\partial X_m^{(0)}(R_\mu)}{\partial t} + \frac{\partial X_m^{(0)}(B)}{\partial a^\mu}\Big](h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu)f - \Big[\Big(\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu}$$

$$- \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\Big)(h_\nu + \varepsilon\Omega^{\nu\mu}W_\mu) - \frac{\partial X_m^{(0)}(B)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial t}\Big]\xi_\mu^0$$

$$+ \frac{\bar{\mathrm{d}}G_M^m}{\mathrm{d}t} + \varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial G_M^m}{\partial a^\alpha}\Big\}.$$

By using the criterion equation (23), and noticing the following equations

$$\Big[\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\Big]h_\nu h_\mu = 0,$$
$$\Big[\frac{\partial X_{m-1}^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\Big]\Omega^{\nu\mu}W_\mu\Omega^{\mu\nu}W_\nu = 0, \tag{31}$$

we can obtain

$$\frac{\tilde{\mathrm{d}}I_{Mz}}{\mathrm{d}t} = \varepsilon^m \Big\{ X_m^{(0)}(R_\mu)\frac{\bar{\mathrm{d}}\xi_\mu^0}{\mathrm{d}t} + X_m^{(0)}(R_\mu)\varepsilon\Omega^{\alpha\beta}W_\beta \frac{\partial \xi_\mu^0}{\partial a^\alpha} - X_m^{(0)}(B)\frac{\bar{\mathrm{d}}f}{\mathrm{d}t} - X_m^{(0)}(B)\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial f}{\partial a^\alpha}$$

$$+ \frac{\partial X_m^{(0)}(R_\mu)}{\partial t}(h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu)f - \frac{\partial X_m^{(0)}(B)}{\partial t} f + \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}(h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu)\xi_\nu^0$$

$$- \frac{\partial X_m^{(0)}(B)}{\partial a^\nu}\xi_\nu^0 - \Big[\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\Big]h_\nu\varepsilon\Omega^{\mu\nu}W_\nu f - \Big[\frac{\partial X_{m-1}^{(0)}(R_\nu)}{\partial a^\mu}$$

$$- \frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\Big]h_\mu\Omega^{\nu\mu}W_\mu f - X_{m-1}^{(0)}(W_\mu)(\xi_\mu^0 - h_\mu f) + X_{m-1}^{(0)}(W_\mu)\varepsilon\Omega^{\mu\nu}W_\nu f$$

$$- \Big[\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\Big]\varepsilon\Omega^{\nu\mu}W_\mu\xi_\mu^0$$

$$+ \Big[\frac{\partial X_{m-1}^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\Big]\Omega^{\nu\mu}W_\mu\xi_\mu^0 + \frac{\bar{\mathrm{d}}G_M^m}{\mathrm{d}t}$$

$$+ \varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial G_M^m}{\partial a^\alpha}\Big\}$$

$$= \varepsilon^m\Big\{\Big[f\frac{\partial X_m^{(0)}(R_\mu)}{\partial t} + \xi_\nu^0\frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\Big](h_\mu + \varepsilon\Omega^{\mu\nu}W_\nu) \tag{32}$$

$$- \Big[f\frac{\partial X_m^{(0)}(B)}{\partial t} + \xi_\nu^0\frac{\partial X_m^{(0)}(B)}{\partial a^\nu}\Big]$$

$$
\begin{aligned}
&- X_m^{(0)}(B)\frac{\bar{\mathrm{d}} f}{\mathrm{d}t} - X_m^{(0)}(B)\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial f}{\partial a^\alpha} + X_m^{(0)}(R_\mu)\frac{\bar{\mathrm{d}}\xi_\mu^0}{\mathrm{d}t} + X_m^{(0)}(R_\mu)\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial \xi_\mu^0}{\partial a^\alpha}\\
&- \left[\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\right] h_\nu\varepsilon\Omega^{\mu\nu}W_\nu f\\
&+ \left[\frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu} - \frac{\partial X_{m-1}^{(0)}(R_\nu)}{\partial a^\mu}\right] h_\mu\Omega^{\nu\mu}W_\mu f\\
&- X_{m-1}^{(0)}(W_\mu)(\xi_\mu^0 - h_\mu f) + X_{m-1}^{(0)}(W_\mu)\varepsilon\Omega^{\mu\nu}W_\nu f\\
&- \left[\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\right]\varepsilon\Omega^{\nu\mu}W_\mu\xi_\mu^0\\
&+ \left[\frac{\partial X_{m-1}^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\right]\Omega^{\nu\mu}W_\mu\xi_\mu^0 + \frac{\bar{\mathrm{d}}G_M^m}{\mathrm{d}t} + \varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial G_M^m}{\partial a^\alpha}\Bigg\}.
\end{aligned}
$$

Using the structure equation (28), we obtain

$$
\begin{aligned}
\frac{\tilde{\mathrm{d}} I_{Mz}}{\mathrm{d}t} &= \varepsilon^m\Bigg\{\left[f\frac{\partial X_m^{(0)}(R_\mu)}{\partial t} + \xi_\nu^0\frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\right]\varepsilon\Omega^{\mu\nu}W_\nu\\
&\quad - \left[f\frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial t} + \xi_\nu^0\frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\right]\Omega^{\mu\nu}W_\nu\\
&\quad - X_m^{(0)}(B)\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial f}{\partial a^\alpha} + X_{m-1}^{(0)}(B)\Omega^{\alpha\beta}W_\beta\frac{\partial f}{\partial a^\alpha} + X_m^{(0)}(R_\mu)\varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial\xi_\mu^0}{\partial a^\alpha}\\
&\quad - X_{m-1}^{(0)}(R_\mu)\Omega^{\alpha\beta}W_\beta\frac{\partial\xi_\mu^0}{\partial a^\alpha} - \left[\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\right]h_\nu\varepsilon\Omega^{\mu\nu}W_\nu f\\
&\quad + \left[\frac{\partial X_{m-1}^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\right]h_\nu\Omega^{\mu\nu}W_\nu f + X_{m-1}^{(0)}(W_\mu)\varepsilon\Omega^{\mu\nu}W_\nu f\\
&\quad - X_{m-2}^{(0)}(W_\mu)\Omega^{\mu\nu}W_\nu f - \left[\frac{\partial X_m^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_m^{(0)}(R_\mu)}{\partial a^\nu}\right]\varepsilon\Omega^{\nu\mu}W_\mu\xi_\mu^0\\
&\quad + \left[\frac{\partial X_{m-1}^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_{m-1}^{(0)}(R_\mu)}{\partial a^\nu}\right]\Omega^{\nu\mu}W_\mu\xi_\mu^0 + \varepsilon\Omega^{\alpha\beta}W_\beta\frac{\partial G_M^m}{\partial a^\alpha} - \Omega^{\alpha\beta}W_\beta\frac{\partial G_M^{m-1}}{\partial a^\alpha}\Bigg\}\\
&= \varepsilon^{z+1}\Bigg\{\left[f\frac{\partial X_z^{(0)}(R_\mu)}{\partial t} + \xi_\nu^0\frac{\partial X_z^{(0)}(R_\mu)}{\partial a^\nu}\right]\Omega^{\mu\nu}W_\nu - X_z^{(0)}(B)\Omega^{\alpha\beta}W_\beta\frac{\partial f}{\partial a^\alpha}\\
&\quad + X_z^{(0)}(R_\mu)\Omega^{\alpha\beta}W_\beta\frac{\partial\xi_\mu^0}{\partial a^\alpha} - \left[\frac{\partial X_z^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_z^{(0)}(R_\mu)}{\partial a^\nu}\right]h_\nu\Omega^{\mu\nu}W_\nu f\\
&\quad + X_{z-1}^{(0)}(W_\mu)\Omega^{\mu\nu}W_\nu f - \left[\frac{\partial X_z^{(0)}(R_\nu)}{\partial a^\mu} - \frac{\partial X_z^{(0)}(R_\mu)}{\partial a^\nu}\right]\Omega^{\nu\mu}W_\mu\xi_\mu^0 + \Omega^{\alpha\beta}W_\beta\frac{\partial G_M^z}{\partial a^\alpha}\Bigg\}.
\end{aligned}
\tag{33}
$$

It shows that $\tilde{\mathrm{d}} I_{Mz}/\mathrm{d}t$ is directly proportional to ε^{z+1}, so I_{Mz} is a zth order adiabatic invariant of the Birkhoffian system.

The adiabatic invariant (29) is a new type of adiabatic invariant. When $z = 0$, the adiabatic invariant (29) will turn into the generalized Mei exact invariant (13) naturally. If we let $f = \tau^0$, (28) and (29) respectively become the structure equation and the Mei adiabatic invariant in [27]. So our results in this paper are more common, and we call the adiabatic invariant (29) as generalized Mei adiabatic invariant for Birkhoffian system. □

5 An Example

The Birkhoffian system is

$$R_1 = a^3, \qquad R_2 = a^4, \qquad R_3 = R_4 = 0, \qquad B = a^2 + \frac{1}{2}[(a^3)^2 + (a^4)^2], \tag{34}$$

and the system is perturbed by the small quantities

$$\varepsilon W_1 = \varepsilon W_2 = 0, \qquad \varepsilon W_3 = \varepsilon a^3, \qquad \varepsilon W_4 = \varepsilon(a^4 + t). \tag{35}$$

Let us study the perturbation to Mei symmetry and adiabatic invariants of the system.

The equations of motion for the system without perturbation are

$$h_1 = \dot{a}^1 = a^3, \qquad h_2 = \dot{a}^2 = a^4, \qquad h_3 = \dot{a}^3 = 0, \qquad h_4 = \dot{a}^4 = -1. \tag{36}$$

Firstly, we seek the exact invariants. Choose infinitesimal generators as

$$\tau^0 = 0, \qquad \xi_1^0 = 1, \qquad \xi_2^0 = t, \qquad \xi_3^0 = \xi_4^0 = 0. \tag{37}$$

It is easy to verify that they satisfy the criterion equation (11), so the infinitesimal generators (37) are Mei symmetrical for the system (34).

From structure equation (12), we obtain

$$\frac{\bar{d}G_M^0}{dt} = f + t\frac{\bar{d}f}{dt}. \tag{38}$$

(i) Let $f = a^1$ we have

$$G_M^0 = \frac{(a^1)^2}{2a^3} + \frac{1}{2}a^3t^2. \tag{39}$$

According to Proposition 1, we can obtain a generalized Mei exact invariant of the system

$$I_{M0} = \frac{(a^1)^2}{2a^3} + \frac{1}{2}a^3t^2 - a^1t = \text{const}. \tag{40}$$

(ii) Let $f = \tau^0 = 0$, we have

$$G_M^0 = a^3 \tag{41}$$

and

$$I_{M0} = a^3 = \text{const}. \tag{42}$$

Equation (42) is a Mei exact invariant of the system.

Then we seek the first order adiabatic invariants. The criterion equation (23) of the perturbation to Mei symmetry gives

$$\left[\frac{\partial X_1^{(0)}(R_\nu)}{\partial a^\mu}-\frac{\partial X_1^{(0)}(R_\mu)}{\partial a^\nu}\right]h_\nu+\left[\frac{\partial X_0^{(0)}(R_\nu)}{\partial a^\mu}-\frac{\partial X_0^{(0)}(R_\mu)}{\partial a^\nu}\right]\Omega^{\nu\mu}W_\mu$$
$$-\frac{\partial X_1^{(0)}(B)}{\partial a^\mu}-\frac{\partial X_1^{(0)}(R_\mu)}{\partial t}-X_0^{(0)}(W_\mu)=0. \tag{43}$$

It has a group of solutions as

$$\tau^1=1,\qquad \xi_1^1=0,\qquad \xi_2^1=a^4,\qquad \xi_3^1=0,\qquad \xi_4^1=-1. \tag{44}$$

From structure equation (28), we obtain

$$\frac{\bar{\mathrm{d}}G_M^1}{\mathrm{d}t}=\left[a^3\frac{\partial f}{\partial a^1}+(a^4+t)\frac{\partial f}{\partial a^2}\right]t-a^3\frac{\partial G_M^0}{\partial a^1}-(a^4+t)\frac{\partial G_M^0}{\partial a^2}+1. \tag{45}$$

(i) Let $f=a^1$ we have

$$G_M^1=\frac{1}{2}a^3t^2-\frac{(a^1)^2}{2a^3}-a^4. \tag{46}$$

According to Proposition 2, the perturbation to Mei symmetry of the system (34) can lead to a first order generalized Mei adiabatic invariant

$$I_{M1}=\frac{(a^1)^2}{2a^3}+\frac{1}{2}a^3t^2-a^1t+\varepsilon\left[\frac{1}{2}a^3t^2-\frac{(a^1)^2}{2a^3}-a^4-t\right]. \tag{47}$$

(ii) Let $f=\tau^0=0$, we have

$$G_M^1=-a^4 \tag{48}$$

and

$$I_{M1}=a^3-\varepsilon(a^4+t). \tag{49}$$

Equation (49) is a first order Mei adiabatic invariant of the system.

Further we can obtain more higher order Mei adiabatic invariants and generalized Mei adiabatic invariants.

6 Conclusion

In this paper, we obtained a generalized Mei adiabatic invariant induced directly from the perturbation to Mei symmetry for Birkhoffian system. If we let $f=\tau^0$, the generalized Mei adiabatic invariant will turn into the Mei adiabatic invariant. And when $z=0$, the generalized Mei adiabatic invariant will turn into the corresponding generalized Mei exact invariant naturally. The form of the generalized Mei adiabatic invariant is general, which has important significance for further study on Mei symmetry and symmetrical perturbation theory.

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